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## A decomposition of the adjoint representation of $U_q(\mathfrak{sl}_2)$

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**Abstract.** A decomposition of the adjoint representation into indecomposable modules for rank one quantum algebras is given. A problem related to the uniqueness of the decomposition is presented.

$U_q(\mathfrak{sl}_2)$

**Definition of  $U_q(\mathfrak{sl}_2)$ .** It is confusing, but there are three objects all of which are called  $U_q(\mathfrak{sl}_2)$ . Thus, first of all, we give the definition of these, and we call them  $U_q^{(l)}$ ,  $U_q^{(m)}$ ,  $U_q^{(s)}$ , respectively.

**DEFINITION.** Let  $K = \mathbb{Q}(q)$  be the field of rational functions.  $U_q^{(l)}$  is the associative algebra over  $K$  defined by the following generators and relations:

Generators are  $e, f, k^{\frac{1}{2}}$  and  $k^{-\frac{1}{2}}$ , relations are

$$\begin{aligned} k^{\frac{1}{2}}ek^{-\frac{1}{2}} &= qe, & k^{\frac{1}{2}}fk^{-\frac{1}{2}} &= q^{-1}f \\ k^{\frac{1}{2}}k^{-\frac{1}{2}} &= k^{-\frac{1}{2}}k^{\frac{1}{2}} = 1, & ef - fe &= \frac{k^2 - k^{-2}}{q^2 - q^{-2}} \end{aligned}$$

**DEFINITION.**  $U_q^{(m)}$  is the subalgebra of  $U_q^{(l)}$  generated by  $e, f, k$  and  $k^{-1}$ .

**DEFINITION.**  $U_q^{(s)}$  is the subalgebra of  $U_q^{(m)}$  generated by  $ek, k^{-1}f, k^2$  and  $k^{-2}$ .

To be more precise,  $U_q^{(m)}$  and  $U_q^{(s)}$  should also be defined by generators and relations, but it is convenient for us to define these as above.

An element  $C = fe + \frac{q^2k^2 + q^{-2}k^{-2}}{(q^2 - q^{-2})^2}$  is called the Casimir element.

**Adjoint action.** These three rank one quantum algebras have natural adjoint action arising from their Hopf algebra structure. Suppose that  $(U, \Delta, S, \epsilon)$  is a Hopf algebra.  $a \in U$  acts on  $U$  as the endomorphism which sends  $x$  to  $\sum a_i^{(1)} x S(a_i^{(2)})$  where,  $\Delta(a) = \sum a_i^{(1)} \otimes a_i^{(2)}$ .

This action is called the adjoint action of  $U$ .

Returning to our case,  $U_q^{(l)}$  has a Hopf algebra structure as follows:  
Its comultiplication is the algebra homomorphism which is uniquely determined by

$$\begin{aligned}\Delta : e &\mapsto e \otimes k^{-1} + k \otimes e \\ f &\mapsto f \otimes k^{-1} + k \otimes f \\ k^{\frac{1}{2}} &\mapsto k^{\frac{1}{2}} \otimes k^{\frac{1}{2}}\end{aligned}$$

Its antipode (which is an antihomomorphism) and its counit are,

$$\begin{aligned}S : e &\mapsto -q^{-2}e, \quad f \mapsto -q^2f, \quad k^{\frac{1}{2}} \mapsto k^{-\frac{1}{2}} \\ \epsilon : e &\mapsto 0, \quad f \mapsto 0, \quad k^{\frac{1}{2}} \mapsto 1\end{aligned}$$

This Hopf algebra structure naturally induces those for  $U_q^{(m)}, U_q^{(s)}$ . Summarizing the above, we have reached the following more concrete definition of the adjoint representation of  $U_q(\mathfrak{sl}_2)$ .

**DEFINITION.**  $U_q^{(l)}$  becomes a  $U_q^{(l)}$ -module by

$$\begin{aligned}Ad(e)x &= exk - q^{-2}kxe \\ Ad(f)x &= fzk - q^2kxf \quad (x \in U_q^{(l)}) \\ Ad(k^{\frac{1}{2}})x &= k^{\frac{1}{2}}xk^{-\frac{1}{2}}\end{aligned}$$

We denote it by  $(Ad, U_q^{ad})$ , and we call it the adjoint representation.

#### BASIC LEMMAS

**Simultaneous eigenvectors for  $Ad(k^{\frac{1}{2}})$  and  $Ad(C)$ .** We start with determining concrete form of simultaneous eigenvectors for  $Ad(k^{\frac{1}{2}})$  and  $Ad(C)$ .

**LEMMA1.** Let  $\overline{K}$  be the algebraic closure of  $K$ . Then any simultaneous eigenvector for  $Ad(k^{\frac{1}{2}})$  and  $Ad(C)$  in  $U_q^{(l)} \otimes \overline{K}$  is either of the following forms (up to non zero scalar):

- (1)  $k^{-2n-m}e^m(1 + \sum_{j=1}^{n-1} a_j^{(m)}(C)k^{2j})(n = 1, 2, \dots)$   
(where,  $a_j^{(m)}(X)$  is a polynomial of degree equal or less than  $j$ .)
- (2)  $k^{-m}e^m$
- (3)  $k^{-2n-m}f^m(1 + \sum_{j=1}^{n-1} a_j^{(m)}(C)k^{2j})(n = 1, 2, \dots)$
- (4)  $k^{-m}f^m$   
( $m = 0, 1, 2, \dots$ )

Using this lemma, one can easily prove the following Lemma2.

**LEMMA 2.**

- (1) There is no highest weight vector whose highest weight is  $q$  to the negative power.
- (2) There is no lowest weight vector whose lowest weight is  $q$  to the positive power.
- (3) Let  $V$  be a submodule of  $U_q^{ad}$ , then  $V \neq 0$  if and only if  $V^0 := V \cap K[k^{\frac{1}{2}}, k^{-\frac{1}{2}}, C] \neq 0$
- (4) Let  $\{V_\alpha\}$  be a set of submodules of  $U_q^{ad}$ , then  $\sum V_\alpha = \oplus V_\alpha$  is equivalent to  $\sum V_\alpha^0 = \oplus V_\alpha^0$

**Concrete submodules.** Now we give definition of certain submodules of  $U_q^{ad}$ .

**DEFINITION.**

- (1)  $V_{half} = \sum_{n,m \in \mathbb{Z}} K[C]k^{n+\frac{1}{2}}e^m + K[C]k^{n+\frac{1}{2}}f^m$
- (2)  $V_{odd} = \sum_{n+m=\text{odd}; n,m \in \mathbb{Z}} K[C]k^n e^m + K[C]k^n f^m$
- (3)  $V_{even} = \sum_{n+m=\text{even}; n,m \in \mathbb{Z}} K[C]k^n e^m + K[C]k^n f^m$

**DEFINITION.**

- (1)  $V_{n+\frac{1}{2}} = Ad(U_q^{(l)})k^{n+\frac{1}{2}} \quad (n \in \mathbb{Z})$
- (2)  $V_{2n+1} = Ad(U_q^{(l)})k^{2n+1} \quad (n \in \mathbb{Z})$
- (3)  $V_{2n} = Ad(U_q^{(l)})C^{-n}k^2 \quad (n \in \mathbb{Z}_{\leq 0})$
- (4)  $V_{2n} = Ad(U_q^{(l)})k^{-n+2}e^n + Ad(U_q^{(l)})k^{-n+2}f^n \quad (n \in \mathbb{Z}_{>0})$

Then, it is easy to see the following.

**PROPOSITION 3.**

- (1)  $U_q^{(l)} = V_{half} \oplus V_{odd} \oplus V_{even}$
- (2)  $U_q^{(m)} = V_{odd} \oplus V_{even}$
- (3)  $U_q^{(s)} = V_{even}$

**lemma for indecomposability.** The next lemma is for proving that  $V_n$ 's are indecomposable as  $U_q^{(*)}$ -module ( $* = l, m, s$ ). But we have to remark that for  $V_{2n}$  ( $n \in \mathbb{Z}_{>0}$ ), we need one more fact that  $Ad(f)^n(k^{-n+2}e^n)$  coincides with  $Ad(e)^n(k^{-n+2}f^n)$  up to scalar.

We can prove this fact by induction on  $n$ .

**LEMMA 4.** Let  $V$  be a submodule of  $U_q^{ad}$ . If  $V^0$  is generated by one element as  $K[Ad(C)]$ -module, and it has no  $Ad(C)$ -eigenvector, then  $V$  is indecomposable.

**PROPOSITION 5.**  $V_n$ 's are indecomposable.

## MAIN RESULT

**Theorem.** To give decomposition of the adjoint representation, it is enough to prove that

**THEOREM.**

$$\begin{aligned} V_{half} &= \bigoplus_{n \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}} V_n \\ V_{odd} &= \bigoplus_{n=\text{odd}} V_n \\ V_{even} &= (\bigoplus_{n=\text{even}} V_n) \oplus V_{soc} \\ (\text{where, } V_{soc} &= \bigoplus K[C]Ad(U_q^{(1)})k^{-n}e^n) \end{aligned}$$

These  $Ad(U_q^{(1)})k^{-n}e^n$  ( $n=0,1,\dots$ ) are irreducible modules. One may call  $V_{soc}$  the socle part of  $U_q^{ad}$  since any irreducible submodule is contained in it.

We can give module structure of these direct summands. Let  $X(n) = U_q^{(1)}/U_q^{(1)}(k^{\frac{1}{2}} - q^n)$ . It is naturally a left module. Then,  $V_{2n}$  ( $n=1,2,\dots$ ) is an amalgamated sum of  $X(n)$  and  $X(-n)$  respectively, and all other summands are isomorphic to  $X(0)$ . Furthermore,  $V_{2n}$  ( $n=0,1,2,\dots$ ) are mutually nonisomorphic.

$V_{half}$  and  $V_{odd}$ . Direct calculation of  $Ad(e)(k^n e^m)$  and  $Ad(f)(k^n f^m)$  shows that  $V_{half} = Ad(U_q^{(1)})(V_{half})^0$  and  $V_{odd} = Ad(U_q^{(1)})(V_{odd})^0$ . Thus it is enough to give decomposition of  $(V_{half})^0$  and  $(V_{odd})^0$  into indecomposable  $K[Ad(C)]$ -modules.

We can show  $C^p k^{n+\frac{1}{2}} \in (V_{half})^0$  and  $C^p k^{2n+1} \in (V_{odd})^0$  by induction on  $p$ .

To prove that  $(V_{odd})^0 = \sum_{n=\text{odd}} V_n^0$ , we introduce a filtration  $\{F_n = \sum_{j \geq i+n} K C^i k^{2j+1}\}$  of  $K[Ad(C)]$ -modules. Then it is easy to see that  $V_{2n+1}^0 \cong F_n/F_{n+1}$ . It completes the proof of the theorem for  $V_{odd}$ . The similar argument is valid for  $V_{half}$ .

$V_{even}$ . The proof of the theorem for  $V_{even}$  splits into two parts. First part is to prove  $(V_{even})^0 = (V_{soc})^0 \oplus (\bigoplus V_{2n}^0)$ . It is the consequence of the following lemma.

**LEMMA 6.** Let  $V^+ = \sum_{j \leq 0} K[C]k^{2j}$ ,  $V^- = \sum_{j > 0} K[C]k^{2j}$ , then

- (1)  $V^+ = \bigoplus_{p,n \geq 0} K C^p Ad(f)^n(k^{-n}e^n)$
- (2)  $V^- = (\bigoplus_{n \leq 0} K[Ad(C)]C^{-n}k^2) \oplus (\bigoplus_{n > 0} K[Ad(C)]k^{2n+2})$
- (3)  $Ad(f)^n(k^{-n+2}e^n) \equiv k^{2n+2}$  up to nonzero scalar modulo  $V^+ \oplus (\bigoplus_{j \leq 0} C^{-j}k^2) \oplus (\bigoplus_{0 < j < n} K[Ad(C)]k^{2j+2})$

Second part is to prove that  $V_{soc} + \sum V_{2n}$  coincides with the whole space  $V_{even}$ . Since  $k^n e^m$  is in the image of  $Ad(e)$  and  $k^n f^m$  is in the image of  $Ad(f)$  if  $n+m \neq 2$ , it is enough to see the following.

## LEMMA 7.

- (1)  $Ad(f)^j(k^{-n-j+2}e^{n+j}) \equiv f_{j,n}(C)k^{-n+2}e^n$  modulo  $ImAd(e)$  where  $f_{j,n}(X)$  is a polynomial of degree  $j$ .  
 (2)  $Ad(e)^j(k^{-n-j+2}f^{n+j}) \equiv f_{j,n}(C)k^{-n+2}f^n$  modulo  $ImAd(f)$ .

## REMARK ON THE UNIQUENESS OF THE DECOMPOSITION

In the previous section, we gave a decomposition of the adjoint representation of  $U_q(sl_2)$  into indecomposable modules. Then it is natural to consider the uniqueness problem of the decomposition up to isomorphism. From this, it arises an interesting problem, which is as follows.

Let  $r, s, n$  be non negative integers such that  $r + s = n$ . Let  $\{p_i\}$  ( $r+1 \leq i \leq n-1$ ) be a set of mutually distinct prime elements of  $K[X]$ . We put

$$I = \{A = (a_{ij}) \in M(n, n, K[X]) \mid a_{ij} \equiv 0 \pmod{p_i \cdots p_{j-1}} (i > r)\}$$

Let  $I^\times$  be the group consisting of invertible elements of  $I$ . Then, what should be natural representatives of  $I^\times \backslash I / I^\times$ ?

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